MONTE CARLO SOLUTION OF A PROBLEM IN SPATIAL FILTRATION

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Difference methods are often applied [1] to liquid filtration subject to complicated boundary conditions. If the pitch is small, this may produce up to 10^{10} algebraic equations. In the usual methods of

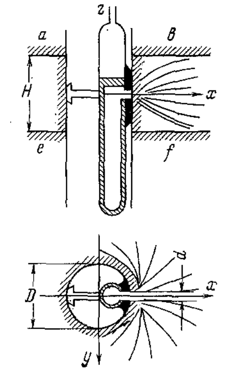

Fig. 1

solution (iteration, elimination of unknowns, etc.) it is necessary to find all solutions, which is very difficult when there are many unknowns and, as a rule, the methods of solution give only slow convergence.

As regards the flow to a sampling hole, it is sufficient to know the pressures at points close to the water surface, so the Monte Carlo method widely used in two-dimensional problems [2] can be applied.

NOTATION

- p—pressure,
- t—time,
- $\varkappa$—pressure-transfer coefficient,
- H—stratum thickness,
- D—well diameter,
- d—flow diameter of samples,
- F—flow area,
- β—compressibility of the liquid,
- $\beta°$—compressibility of the porous medium,
- β^*—reduced compressibility,
- m—porosity factor,
- k—permeability,
- W—no. of steps of wandering particle,
- μ—absolute viscosity,
- $D(\xi)$—variance estimator,
- m_k—no. of entries to well in 100 tests,
- m_0—no. of entries to well in 1000 tests,
- ε—absolute error,
- δ—relative error.

Let N be the number of tests and N_i the number of times the wandering particle enters the flow, with n_ν the number of entries from initial points located on elementary areas F_ν (ν = 1, 2, 3, 4) and ω a random number.

§1. Consider the actual geometry of the transient flow to the sampler on a logging cable and in a stratum of limited thickness but infinite extent (Fig. 1). The pressure distribution is [3] as follows for a homogeneous isotropic elastic stratum subject to filtration of a homogeneous liquid:

$$\nabla^2 p = \frac{\partial p}{\varkappa \partial t}, \quad \varkappa = \frac{k}{\mu(m\beta+\beta°)}, \quad \tau = \varkappa t, \quad \nabla^2 p = \frac{\partial p}{\partial \tau}, \qquad (1.1)$$

while the condition of impermeability at the boundaries ab and ef is

$$\partial p / \partial z = 0 \quad \text{for } z = \pm {}^1\!/_2 H, \ {}^1\!/_4 D^2 \leqslant x^2 + y^2 < \infty$$

the condition of impermeability of the clay lining of the well (diameter d) and of the sealing unit is

$$\partial p / \partial x = \partial p / \partial y = 0 \quad \text{for } x^2 + y^2 = {}^1\!/_4 D^2, \ z^2 + y^2 \geqslant {}^1\!/_4 d^2,$$

the initial condition is

$$p(x, y, z, t)|_{t=0} = p° \quad \text{for } x^2 + y^2 \geqslant 1/4\ D^2,$$

the pressure in the vessel after the start of the flow is

$$p = p_0(t) \quad \text{for } t > 0,$$
$$x^2 + y^2 = {}^1\!/_4 D^2, \ z^2 + y^2 < {}^1\!/_4 d^2, \ x > 0,$$

and the flow through the sampler surface is [4]

$$q(t) = \int_F \frac{k}{\mu}\frac{dp}{dn}(t)\, dF, \qquad \frac{k}{\mu} = \varkappa(m\beta + \beta°) = \varkappa\beta^*. \qquad (1.2)$$

We replace the differential equation by a difference one with a rectangular net:

$$\nabla^2 p\, \frac{\partial p}{\partial \tau}, \qquad \frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$

$$\frac{p(x, y, z, \tau + h_\tau)}{h_\tau} =$$

$$= \frac{p(x + h_x, y, z, \tau) - 2p(x, y, z, \tau) + p(x - h_x, y, z, \tau)}{h_x^2} +$$

$$+ \frac{p(x, y + h_y, z, \tau) - 2p(x, y, z, \tau) + p(x, y - h_y, z, \tau)}{h_y^2} +$$

$$+ \frac{p(x, y, z + h_z, \tau) - 2p(x, y, z, \tau) + p(x, y, z - h_z, \tau)}{h_z^2}.$$

We put $p(x, y, z, \tau) = p$ and take a uniform net $h_x = h_y = h_z = h$:

$$p(\tau + h_\tau) = \left(1 - \frac{6h_\tau}{h^2}\right) p +$$

$$+ p(x + h)\frac{h_\tau}{h^2} + p(x - h)\frac{h_\tau}{h^2} + p(y + h)\frac{h_\tau}{h^2} +$$

$$+ p(y - h)\frac{h_\tau}{h^2} + p(z + h)\frac{h_\tau}{h^2} + p(z - h)\frac{h_\tau}{h^2}.$$

We put $h_\tau/h^2 = \alpha$; then

$$p(\tau + h_\tau) = (1 - 6\alpha)\ p +$$

$$+ \alpha\ [p(x + h) + p(x - h) + p(y + h) +$$

$$+ p(y - h) + p(z + h) + p(z - h)].$$

We take the relation between the time scale and the pitch of the net such that $1 - 6\alpha = 0$, so $\alpha = h_\tau/h^2 = 1/6$ and $h_\tau = h^2/6$,

$$p(\tau + h_\tau) = {}^1\!/_6\, [p(x + h) + p(x - h) + p(y + h) +$$
$$+ p(y - h) + p(z + h) + p(z - h)].$$

The solution may thus be realized as a random process with probability 1/6.

The plane $z = \pm {}^1\!/_2 H, \ {}^1\!/_4 D^2 \leqslant x^2 + y^2 < \infty$

$$\frac{\partial p}{\partial z} = 0, \quad \frac{p(z) - p(z - h)}{h} = 0,$$

The cylindrical surface of the borehole $x^2 + y^2 = \frac{1}{4} D^2$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \quad \frac{p(x) - p(x-h)}{h} = 0,$$

$$\frac{p(y) - p(y-h)}{h} = 0,$$

excluding the part of the surface $y^2 + z^2 = d^2/4$, $x = 0$, where $p(x, y, z, \tau) = p_0(\tau)$.

The initial point

$$p(\tau = 0) = p^\circ \quad \text{for } x^2 + y^2 = \frac{1}{4} D^2.$$

Here $p(z) = p(z - h)$ is reflection from the boundaries of the stratum and $p(x) = p(x - h)$, $p(y) = p(y - h)$ is reflection from the surface of the well.

§2. To find the pressure at any point by this method it is necessary to perform a sufficient number of random walks from that point in order to find the mathematical expectation for the capture probability, which will [5] be an approximate value for the solution for that point. The following rules are used. At each node we perform a random

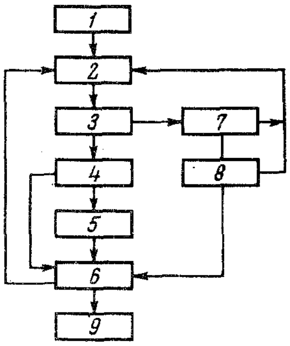

Fig. 2

choice, which gives the point to which the wandering particle must pass. If this operation at a special node on the runoff surface with a given pressure leads to its stopping there, this capture is recorded, and the random walk begins again from the initial point. If the random walk takes the particle through the special nodes at the boundaries of the stratum and the surface of the well, then the point is reflected. If the particle has not entered the runoff after W steps, the stratum pressure is calculated (the pressure in the sampler is calculated in the converse case). This process is repeated a sufficient number of times, and the number of captures is divided by the number of repetitions to get the approximate value of the solution for the node from which all the random walks began:

$$P_i(\tau) = \frac{N_1}{N} p_0(\tau) + \left(1 - \frac{N_i}{N}\right) p^\circ. \qquad (2.1)$$

Figure 2 shows the block diagram for the process; 1 introduces the program and the initial data; 2 generates the random number ω, de-

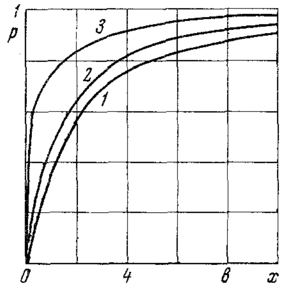

Fig. 3

fines one of the six possible directions, and produces the random walk in the corresponding spatial coordinate; 3 tests for entry to the boundary region and produces reflection; 4 tests for entry to the runoff; 5 calculates the pressure in the vessel; 6 monitors the number of tests; 7 pro-

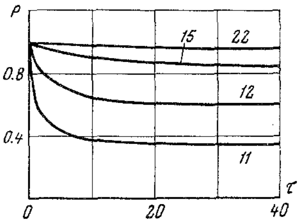

Fig. 4

duces time stepping and monitors τ; 8 detects that the particle has not entered the runoff and calculates the stratal pressure; and 9 derives the mathematical expectation from (2.1).

A standard program for the M-20 computer has been compiled on this basis, and the following calculations have been performed.

1) The pressure distribution along the x-axis, along a generator of the well, and along the perimeter in the xy-plane (Fig. 3). The calculations were performed for H = 200 cm, D = 20 cm, $\tau = 200$ cm^2, h = 0.5 cm, N = 1000, $p_0(\tau) = 0$, $p^\circ = 1$.

The variation along the x-axis (curve 1) is comparatively slow, being more rapid along the generator (curve 2) and most rapid along the perimeter (curve 3).

2) Points on the x-axis have the following dependence of pressure p on time.

$\tau=3$	6	9	12	15	18	21	$(x=11)$
$p=0.44$	0.39	0.38	0.37	0.36	0.35	0.35	
$\tau=5$	10	15	20	25	30	35	$(x=12)$
$p=0.67$	0.63	0.61	0.60	0.59	0.59	0.59	
$\tau=20$	40	60	80	100	120	140	$(x=15)$
$p=0.86$	0.84	0.83	0.82	0.81	0.81	0.80	
$\tau=50$	100	150	200	250	300	350	$(x=22)$
$p=0.97$	0.96	0.95	0.94	0.94	0.94	0.94	

Figure 4 has been constructed from these results. For x = 11 and x = 12, the pressure reaches the asymptotic value for $\tau = 30$ cm^2.

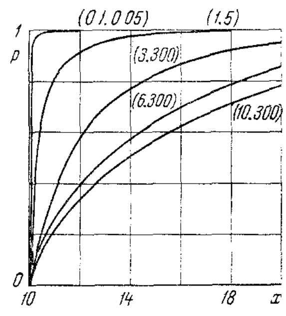

Fig. 5

3) The pressure distribution along the x-axis was calculated for (d, τ) pairs as shown in Fig. 5. The diameter of the depression funnel increases with the diameter of the sampler inlet.

§3. Expression (1.2) contains the pressure gradient along the normal to the flow surface, which was determined approximately in calculation of the liquid flow as the ratio of the pressure at some point to the distance of the point from the flow surface, the pressure being given by (2.1). As it was assumed that the filtration geometry affects the flow rate, 13 points were taken at 0.3 cm from the surface with a disposition as shown in Fig. 6, which were used in calculating the mean pressure gradients. The liquid flow rate was then calculated as the sum of the rates of flow through the individual areas.

The calculations were carried out as follows. The pressure gradient is approximately

$$\frac{dp}{dn}(\tau) \approx 3.33\,[p_i(\tau) - p_0(\tau)] \quad (i = 1, 2, 3, \ldots, 13). \qquad (3.1)$$

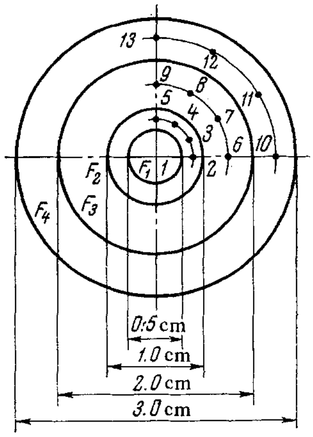

Fig. 6

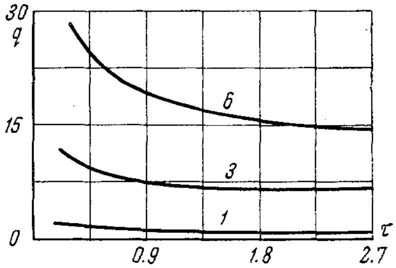

Fig. 7

τ, cm²	$q_j(\tau)$, cm³/sec	
	$D=20$	$D=60$
0.01	26.75	26.75
0.2	11.59	11.59
0.4	9.33	9.33
0.6	8.33	8.33
0.8	7.79	7.79
1.0	7.43	7.43
1.2	7.17	7.17
1.4	6.96	6.96
1.6	6.82	6.82
1.8	6.69	6.69
2.0	6.61	6.61

τ=0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	(d=1)
q=1.82	1.67	1.60	1.56	1.54	1.51	1.50	1.50	1.48	1.47	
τ=0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	(d=3)
q=11.59	9.33	8.33	7.79	7.43	7.17	6.96	6.82	6.69	6.61	
τ=0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0	(d=6)
q=28.17	21.73	19.11	17.50	16.54	15.75	15.11	14.55	14.34	14.18	

The flow rate through the surface for each time interval is

$$q_j(\tau) = 3.33\varkappa\beta^* \Big\{[p_1(\tau) - p_0(\tau)] F_1 + \left[\frac{1}{4}\sum_{i=2}^{5} p_i(\tau) - p_0(\tau)\right] F^2 + \left[\frac{1}{4}\sum_{i=6}^{9} p_i(\tau) - p_0(\tau)\right] F_3 + \left[\frac{1}{4}\sum_{i=10}^{13} p_i(\tau) - p_0(\tau)\right] F_4\Big\}$$

$$(j = 1, 2, 3, \ldots, 11),$$

$$F = 2.25\ \pi, \quad F_1 = 0.0625\ \pi, \quad F_2 = 0.1875\ \pi,$$

$$F_3 = 0.75\ \pi, \quad F_4 = 1.25\ \pi\ . \tag{3.2}$$

The mean pressure gradient on each area for $p_0(\tau) = 0$ is given by (2.1) and (3.1) by

$$p_1(\tau) = \left(1 - \frac{Ni}{N}\right) p^\circ,$$

$$\frac{1}{4}\sum_{i=2}^{5} p_i(\tau) = \left(1 - \frac{1}{4N}\sum_{i=2}^{5} N_i\right) p^\circ,$$

$$\frac{1}{4}\sum_{i=6}^{9} p_i(\tau) = \left(1 - \frac{1}{4N}\sum_{i=6}^{9} N_i\right) p^\circ,$$

$$\frac{1}{4}\sum_{i=10}^{13} p_i(\tau) = \left(1 - \frac{1}{4N}\sum_{i=10}^{13} N_i\right) p_0\ ,$$

where

$$n_1 = N_1,$$

$$n_2 = \frac{1}{4}\sum_{i=2}^{5} N_i, \quad n_3 = \frac{1}{4}\sum_{i=6}^{9} N_i, \quad n_4 = \frac{1}{4}\sum_{i=10}^{12} N_i\ . \tag{3.3}$$

It is more convenient to operate with the n of (3.3) than with the mathematical expectation of (2.1); then (3.2), with allowance for the F_ν (ν = 1, 2, 3, 4) becomes

$$q_j(\tau) = 0.05083\ \pi\varkappa\beta^* \{36\ p^\circ - p^\circ/N\ [n_1 + 3n_2 + 12n_3 + 20n_4]\}\ . \tag{3.4}$$

Results from (3.4) are given below.

1. Effects of D. Calculations were made for D of 20 and 60 cm with $\varkappa = 10^5$ cm²/sec, H = 200 cm, d = 3 cm, $\beta^* = 3 \cdot 10^{-5}$ cm²/kg, $p^\circ = 1$ kg/cm², $p_0(\tau) = 0$, N = 1000, and h = 0.1 cm (table).

The results show that the flow rate is almost independent of D (the difference lies at the limit of error of the calculation), because there is no relation between D and the pressure gradient.

2) Effects of H. The flow rate was calculated for H of 10, 20, 40, 80, and 200 cm, with the other parameters the same; the effect of H was very slight, the flow rate q = 6.612 cm³/sec being as for τ = 2 cm², i.e., the difference was less than the error. This occurs mainly because the depression funnel extends only a small way along the generator (Fig. 3), so H influences q only when H is comparable with d.

3) Effects of d. Figure 7 gives results for d of 1, 3, and 6 cm with the other parameters as above.

The results reveal a nearly linear dependence of q on d, the approach to the asymptote being the more rapid the smaller d.

§4. There are two main sources of error in calculating the pressure gradient: replacement of the differential equation by a difference equation and use of the Monte Carlo method.

1) By analogy with the Runge-Kutta principle, the error of approximation for (1.1) is [6]

$$|R| \leqslant h^2 (1/_{12}\ M_2 - 1/_4\ M_1)\ . \tag{4.1}$$

The error includes second- and fourth-order derivatives. As an analytic expression for the function is not available, these derivatives can be found only approximately. Large errors arise in replacing fourth-order derivatives by difference relations, and so the result does not reflect the true error of approximation, so the following approach was used. The pressure was calculated with steps of h and 2h (0.706, 0.743) at a given point, which gives the value of the function at the point with an estimate of the error:

$$|R_1| \leqslant h^2 (1/_{12}\ M_2 - 1/_4\ M_1),$$

$$|R_2| \leqslant (2\ h)^2 (1/_{12}\ M_2 - 1/_4\ M_1).$$

The maximum error for a step of 0.5 cm is

$$1/_{12}\ M_2 - 1/_4\ M_1 = 0.037.$$

This is substituted into (4.1) to give $|R| \leq 0.0093$, so $\delta \leq 1.3\%$ for the pressure gradient at the surface.

3) The error of the Monte Carlo method is estimated with a probability of 0.9 as

$$\varepsilon = 1.64\ \sqrt{0.1 D(\xi)},$$

$$D(\xi) = \frac{(\xi_1 - \xi^\circ)^2 + (\xi_2 - \xi^\circ)^2 + \cdots + (\xi_{10} - \xi^\circ)^2}{10 - 1},$$

$$\xi^\circ = 0.001 m_0, \quad \xi_k = 0.01 m_k \quad (k = 1, 2, 3, \ldots, 10)\ . \tag{4.2}$$

The number of entries to the exit surface was calculated per 100 and 1000 cycles, which from (4.2) gives $\varepsilon = 0.045$ and $\delta = 6.4\%$.

The relative error of the calculation thus does not exceed 7.7%.

REFERENCES

1. G. G. Vakhitov, "Solution of problems in underground hydraulics by the method of finite differences," Trudy VNII, Gostopekhizdat, no. 10, 1957.

2. M. I. Shvidler, "Solution of two-dimensional filtration problems by the Monte Carlo method," Izv. AN SSSR, Mekhanika i mashinostroenie, no. 1, 1963.

3. V. N. Shchelkachev, "The basic equations for the motion of an elastic liquid in an elastic porous medium," Dokl. AN SSSR, 52, no. 2, 1946.

4. L. S. Leibenzon, Collected Works, Vol. 2 [in Russian], Izd. AN SSSR, 1953.

5. N. P. Buslenko and Yu. A. Shreider, The Monte Carlo Method [in Russian], Fizmatgiz, 1961.

6. I. S. Berezin and N. P. Zhidkov, Computational Methods, Vol. 2 [in Russian], Fizmatgiz, 1960.

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